LAPLACE TRANSFORM:
Step Function and Impulse Function

LECTURE 10#

INTRODUCTION

This section is important since we shall reach the point where Laplace Transform method shows its superiority in engineering applications.

We will introduce two functions:

1. Unit step function: → $u(t)$

2. Unit impulse function (dirac's delta): → $\delta(t)$

These functions are suitable for solving ODEs with complicated right sides of considerably engineering interest such as single waves, input (driving forces) that are discontinuous or act for some time only.

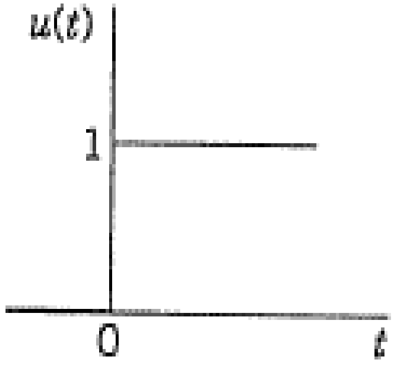
1. UNIT STEP FUNCTION

Definition

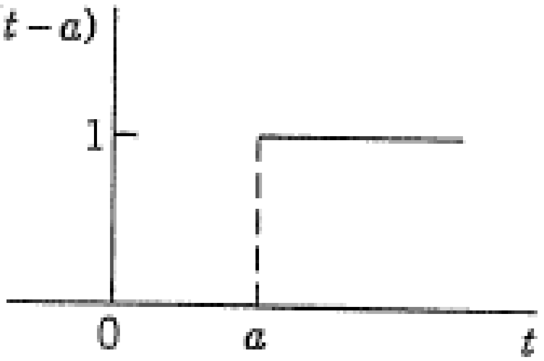
By definition, unit step function, $u(t)$, has a sudden jump to 1 at t=0 and remains 1 for t>0.

Similarly, $u(t\text{-}a)$ is a right shifted or delayed version of $u(t)$ by amount of a (unit time).

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \qquad u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$



Unit step function $u(t)$

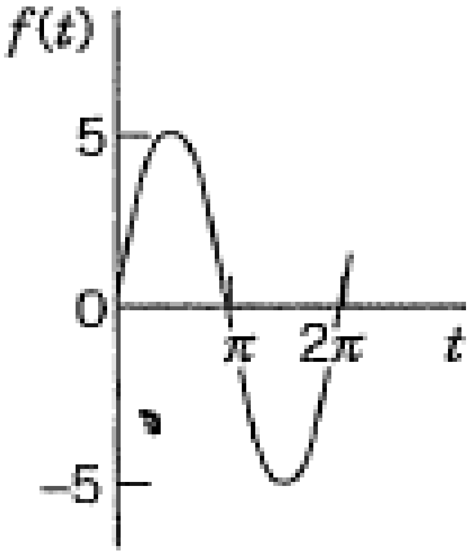


Unit step function $u(t-a)$

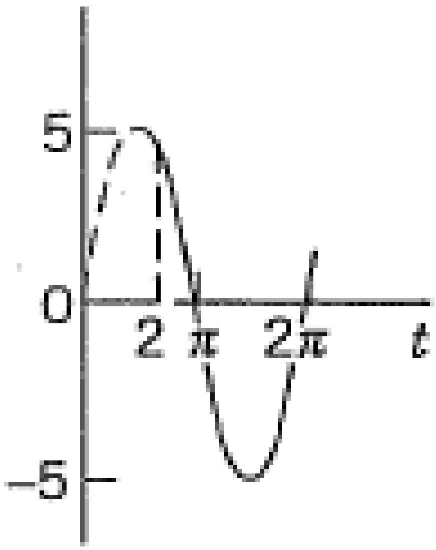
1. UNIT STEP FUNCTION

Effect of Unit Step Function

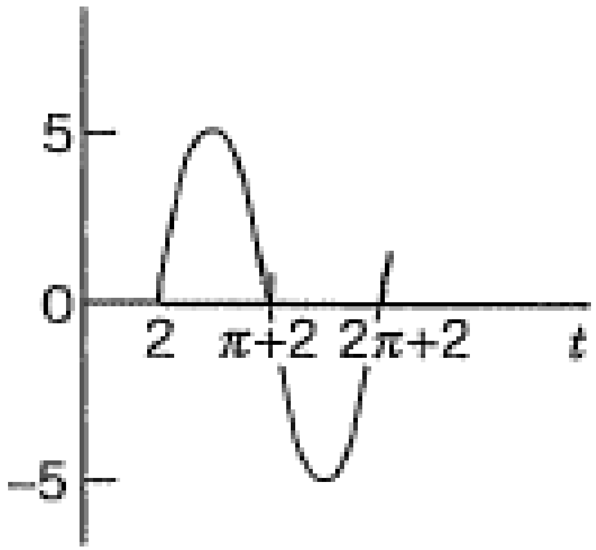
*Let $f(t) = 0$ for all negative t. Then $f(t-a)u(t-a)$ with $a > 0$ is $f(t)$ **shifted** (translated) to the right by the amount a.*



(A). $f(t)=5\sin t.u(t)$



(B). $5\sin t.u(t-2)$



(C). $5\sin(t-2).u(t-2)$

1. UNIT STEP FUNCTION

Laplace of Unit Step Function

Laplace transform of $u(t)$ is given by:

$$\ell\{u(t)\} = \int_0^\infty u(t)e^{-st}\,dt = \int_0^\infty 1.e^{-st}\,dt = \frac{1}{s}$$

For $u(t\text{-}a)$:

$$\ell\{u(t-a)\} = \int_0^\infty u(t-a)e^{-st}\,dt$$

$$= \int_0^a u(t-a)e^{-st}\,dt + \int_a^\infty u(t-a)e^{-st}\,dt$$

$$= 0 + \int_a^\infty 1.e^{-st}\,dt = \frac{-1}{s}\left[e^{-\infty} + e^{-as}\right] = \frac{e^{-as}}{s}$$

SECOND SHIFTING THEOREM

Previously, the first shifting Theorem (s-shifting) is:

$$\ell\left\{e^{at} f(t)\right\} = F(s-a) \quad \Longleftrightarrow \quad e^{at} f(t) = \ell^{-1}\left\{F(s-a)\right\}$$

Laplace

Inverse Laplace

Now, we have second shifting Theorem (t-shifting):

$$\ell\left\{f(t-a)u(t-a)\right\} = e^{-as} F(s) \quad \Longleftrightarrow$$

Laplace

$$\ell^{-1}\left\{e^{-as} F(s)\right\} = f(t-a)u(t-a) = f(t)u(t)\Big|_{t \to t-a}$$

Inverse Laplace

APPLICATION OF SECOND SHIFTING THEOREM

EXAMPLES 1-2:

1. Find the Laplace Transform of the function:

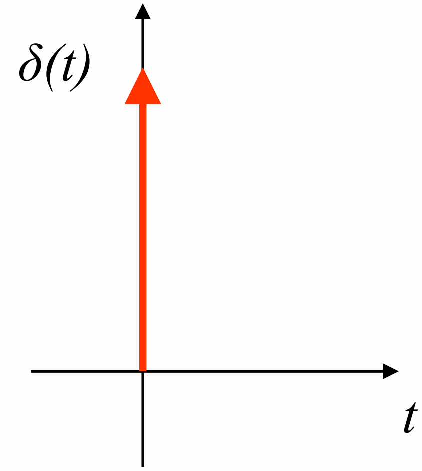
$$f(t)=\begin{cases} 2, & 0<t<\pi \\ 0, & \pi<t<2\pi \\ \sin t, & t>2\pi \end{cases}$$

2. Find the inverse Laplace transform of:

$$F(s)=\frac{2}{s^2}-\frac{2e^{-2s}}{s^2}-\frac{4e^{-2s}}{s}+\frac{se^{-\pi s}}{s^2+1}$$

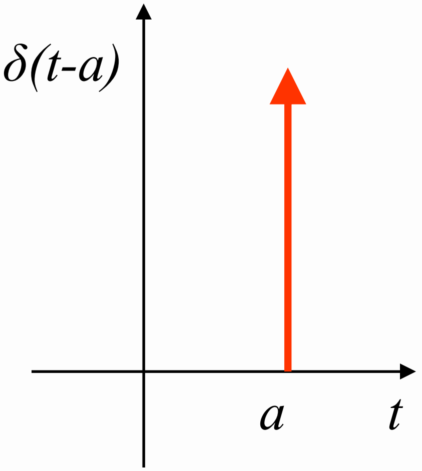
2. UNIT IMPULSE (Dirac's Delta) FUNCTION

The Dirac's delta can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite.



$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$ and $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Similarly, for the shifted case:

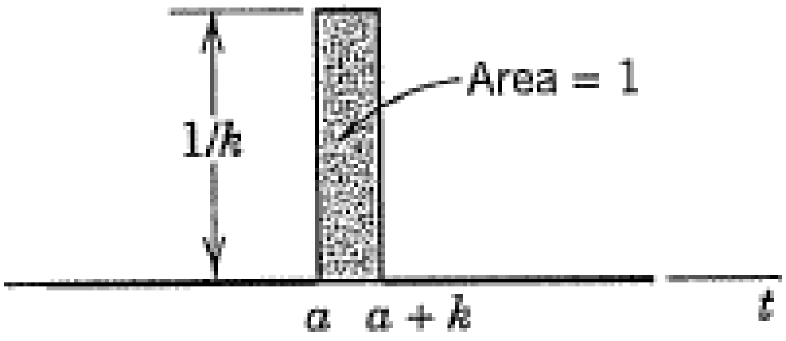


$$\delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & t \neq a \end{cases}$$

2. UNIT IMPULSE (Dirac’s Delta) FUNCTION

Phenomena of an impulsive nature such as action of forces, voltage over short intervals of time, a ship is hit by a single wave and so on, can be modeled by Dirac’s delta and can be solved efficiently by Laplace transform.

$$f_k(t-a)=\begin{cases}1/k & \text{if } a \leqq t \leqq a+k\\ 0 & \text{otherwise}\end{cases}$$



The function $f_k(t-a)$

$$\int_{-\infty}^{\infty} f_k(t-a)\,dt = \int_{a}^{a+k} \frac{1}{k}\,dt = 1$$

Note $: k \to 0$ (small)

2. UNIT IMPULSE (Dirac's Delta) FUNCTION

Laplace of Unit Impulse Function

To get the Laplace Transform of impulse function, $\delta(t-a)=f_k(t-a)$, we represent it in terms of two step functions:

$$f_k(t-a)=\frac{1}{k}\left[u(t-a)-u(t-(a+k))\right]$$

$$\ell\left\{f_k(t-a)\right\}=\frac{1}{ks}\left[e^{-as}-e^{-(a+k)s}\right]$$

For k is small:

$$\ell\left\{f_k(t-a)\right\}=\ell\left\{\delta(t-a)\right\}=\lim_{k\to 0}e^{-as}\left[\frac{1-e^{-ks}}{ks}\right]$$

$$\boxed{\ell\left\{\delta(t-a)\right\}=e^{-as} \qquad \text{and} \qquad \ell\left\{\delta(t)\right\}=1}$$

APPLICATION TO PHYSICAL SYSTEMS

EXAMPLE 3:

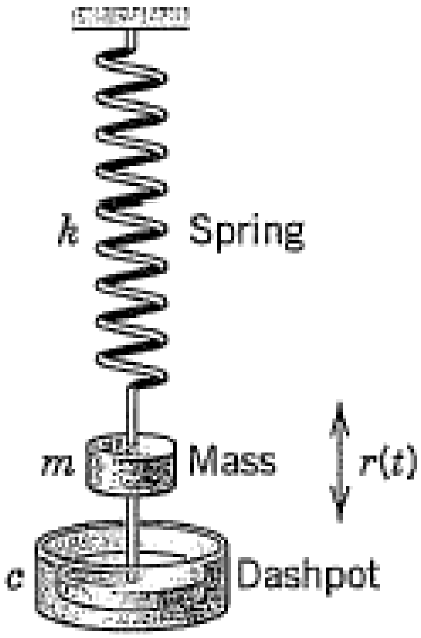
Determine $y(t)$ which is the response of the damped mass-spring system due to input $r(t)$. $y(t)$ is horizontal mass position at time instance. The system is modeled by:

$$\ddot{y} + 3\dot{y} + 2y = r(t); \qquad \dot{y}(0) = y(0) = 0$$

When $r(t)$ is:

a). $r(t)=u(t-1)-u(t-2)$ →square wave

b). $r(t)=\delta(t-1)$ →unit impulse



EXERCISES 1-2:

Solve the followings:

1. $\dot{y} + 2y = r(t); \quad y(0) = 0$

   $where:$

   $$r(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

2. $\ddot{y} + 4y = \sin t . u(t - 2\pi); \qquad y(0) = 1 \;\&\; \dot{y}(0) = 0$